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AN ANALYTICAL SOLUTION FOR THE NONLINEAR INVERSE CAUCHY PROBLEM

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Abstract. This paper discusses the recovering of both Dirichlet and Neumann data on some part of the domain boundary, starting from the knowledge of these data on another part of the boundary for a family of quasi-linear inverse problems. The nonlinear problem is reduced to a linear Cauchy problem for the Laplace equation coupled with a sequence of nonlinear scalar equations. We solve the linear problem using a closed-form regularization analytical solution. Various numerical examples and effects of added small perturbations into the input data are investigated. The numerical results show that the method produces a stable reasonably approximate solution.

Keywords: Nonlinear inverse problems, Cauchy problem, analytical regularization solution.

AMS Subject Classification: 65N21, 65Y05.

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1 Introduction

In the present work, we are interest by the resolution of a class of nonlinear inverse boundary value problems which describes numerous applications in many area of science and engineering, see Choulli (2009); Essaouini et al. (2004a,b); Isakov (2017); Kabanikhin (2012); Lavrent'ev (2013); Qiu et al. (2020) and the references therein. As examples we cite: the determination of the surface fux and/or surface temperature when the measurement of the surface temperature are extremely difficult due to hostile environment such as high temperature, erosion or surface sublimation. For this class of problems, one boundary condition is not specified. Instead, some measurement are given at some locations of the system. In order to overcome the difficulties mentioned above, we give an efficient numerical method for solving this kind of inverse problem. A transformation of the governing equation is used such that all the nonlinear aspect of the problem are transferred to the boundary of the domain. This reduces the nonlinear inverse problem to solve a Laplace equation with insufficient Dirichlet-Neuman mixed boundary conditions. That allowed us to derive exactly an analytical regularization solution to approach the real solution. The solution is simply calculated and does not require any domain discretization nor linear system resolution and is therefore very efficient.

2 Formulation of the problem and the description of the method

In this work we are concerned by the following class of nonlinear inverse boundary problem:

$$\begin{cases} -\nabla q(T)\nabla T & = & 0 & \text{on } \Omega \\ T \mid_{\Gamma_d} & = & f_d & \text{on } \Gamma_d \\ T \mid_{\Gamma_2} & = & f_2 & \text{on } \Gamma_2 \\ q(T)\partial_\nu T \mid_{\Gamma_2} & = & g_2 & \text{on } \Gamma_2 \\ q(T)\partial_\nu T \mid_{\Gamma_n} & = & g_n & \text{on } \Gamma_n \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain for which the boundary Γ is such that

$$\Gamma = \Gamma_0 \cup \Gamma_d \cup \Gamma_2 \cup \Gamma_n.$$

We denote by ν the outward normal vector to $\partial\Omega$ and by ∂_ν the normal derivative operator. There is no boundary conditions prescribed on Γ_0. The real function K is non negative.

Note that, in the modeling of inverse boundary problem in heat transfer, $q(T)$ is a conductivity coefficient, T is the temperature distribution within the system Ω and Γ_0 is the inaccessible boundary.

To illustrate our procedure, let us introduce the transformed variable ω which satisfies

$$\nabla\omega = q\,(T)\,\nabla T$$

this implies that the governing equation becomes the Laplace equation and the problem (1) is reduced to solve the following linear Cauchy problem:

$$\begin{cases} -\Delta\omega & = & f & \text{in } \Omega \\ \omega & = & F(f_d) & \text{on } \Gamma_d \\ \omega & = & F(f_2) & \text{on } \Gamma_2 \\ \partial_\nu\omega & = & g_2 & \text{on } \Gamma_2 \\ \partial_\nu\omega & = & g_n & \text{on } \Gamma_n \end{cases} \tag{2}$$

followed by a serie of nonlinear equations

$$F\,(T\,(X)) = \omega\,(X) \quad \forall\, X \in \Omega, \tag{3}$$

where $\omega\,(X)$ is the value of solution of (2) at a point X in $\bar{\Omega}$ and F denotes the used transformation formula defined by

$$F\,(T) = \int_0^T q\,(t)\,dt. \tag{4}$$

Note that (2) is the classical linear Cauchy problem for the Laplace equation. To obtain the solution of the initial problem (1), one have to solve (2) using any method dedicated to solving the linear Cauchy problem then solve the scalar nonlinear equation (3) by any numerical method for solving the nonlinear equations. it should be noted that the resolution of the equation (3 for an $X \in \Omega$ is independent of any discretization of the domain and that the resolution for different points from omega can be done in a parallel way, the method is therefore a method without mesh. We will in the following use a method, which preserves this character, to solve the problem of Cauchy (2).

It is important to underline that although the problem (2) may have a unique solution, it is well known that this solution is unstable with respect to small perturbations on the boundary data it is severely ill-posed (Hadamard, 1953; Alessandrini et al., 2009; Belgacem, 2007).

This inverse problem has been intensively investigated by several researchers over the last half century.

One of the first methods introduced as an approximation of the Cauchy problem is the quasi-reversibility method (Latté & Lions, 1969). Since then it has been widely applied in different fields and several variants have been developed, one can mention for example Bourgois (2006); Klibanov & Santosa (1991). Other methods, based on formulations in the form of an optimal control problems where a functional taking into account one of the conditions on the overdetermined part, have been developed Andrieux et al. (2006); Chakib & Nachaoui (2006); Kabanikhin & Karchevsky (2012). One of the most popular techniques is that of Tikhonov regularization (Berntsson et al., 2017; Cimetière et al., 2001; Chang et al., 2001; Kabanikhin & Karchevsky , 2012; Kabanikhin et al., 2013; Liu & Wei, 2013). Among the many numerical methods, the schemes based on iteration have also been developed previously by Kozlov et al. (1991), Jourhmane & Nachaoui (1996, 1999, 2002). The relaxation JN method introduced in the last papers drastically reduced the number of iterations required to achieve convergence. It was used in elasticity (Ellabib & Nachaoui, 2008; Marin & Johansson, 2010), and recently for Cauchy problem governed by Stocks equation (Chakib et al., 2018) and for the Helmholtz equation (Berdawood et al., 2020, 2021; Berdawood-Nachaoui et al., 2021). This relaxation JN method has been applied the to simulate the two-dimensional non linear elliptic problem (Essaouini et al., 2004a,b) and recently (Aboud et al., 2021) combined the domain decomposition method and the relaxation method to solve a Cauchy problems on inhomogenious material. In Nachaoui et al. (2021a), efficient iterative domain decomposition like-methods was developed.

In contrast to the methods mentioned above, direct and/or mesh-free methods have been developed over the last two decades. Their efficiencies are independent of the shape of the mesh. Some mesh-free methods can be used in arbitrary fields, for example the collocation techniques together with the expansions by different basis-functions (Hu et al., 2005; Kuo et al., 2013; Li et al., 2008; Nachaoui et al., 2018; Shen, 2002; Tian et al., 2008) Other methods have been developed for solving Cauchy's problems. The reader can consult for example Bergam et al. (2019); Berntsson et al. (2017); Ellabib et al. (2021); Isakov (2017); Juraev (2019, 2020); Nachaoui (2003, 2004) and the references therein.

In the following we use the method introduced in Liu (2011). Its particularity is that it provides an analytical regularization solution which can be implemented without resorting to iteration. (Nachaoui et al., 2021b) have shown that this analytical method is highly effective. They showed that the method can easily be parallelized.

Note that the nonlinear Cauchy problem (1) has been solved in Essaouini et al. (2004a,b) using an iterative procedure based on a method by Jourhmane & Nachaoui (2002) for linear inverse problem combined with a dynamically estimated relaxation parameter in order to increase the rate of convergence. The solution of problem (1) for $(x, y) \in \overline{\Omega}$ has to be evaluated only after a stopping criterion has been satisfied. This was done by solving the nonlinear scalar equation using the Newton-Raphson procedure. Despite the fact that this procedure is simple to implement, it requires discretization and the resolution of several linear systems. The resolution of the scalar nonlinear equation is solved using the linear Cauchy problem approximation.

We propose here the use of an analytical solution for the resolution of the linear Cauchy problem (2) which preserves the character of parallelization induced by the decoupling (2 - (3) We show that this allows to reduce the resolution of the nonlinear Cauchy problem to the resolution of a scalar nonlinear equation for each point of domain of study.

3 Description of the analytical method

3.1 Solution of the linear Cauchy problem

Let us consider $\Omega =]0, L[\times]0, b[$ with its boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where $\Gamma_0 =]0, L[\times\{b\}$, $\Gamma_1 =]0, L[\times\{0\}$, $\Gamma_2 = \{0\}\times]0; b[$ and $\Gamma_3 = \{L\}\times]0; b[$. The Cauchy problem for the Laplace equation that will be considered is as follows:

$$\Delta u(x,y) = 0, \text{ in } \Omega, \tag{5}$$

$$u(x,y) = 0, \quad \text{on } \Gamma_2 \cup \Gamma_3 \tag{6}$$

$$u(x,y) = 0, \quad \text{on } \Gamma_0 \tag{7}$$

$$\partial_y u(x,y) = h(x), \quad \text{on } \Gamma_0, \tag{8}$$

where $h(x)$ is a given function. Problem (5)-(8) is a theoretical model describing some real phenomenon. The model clearly requires that $h(x)$ belongs to a certain class M of functions where problems (5)-(8), have a solution, and it makes sense to extend that solution over Γ_1 in the same sense in which boundary conditions over Γ_0 are given.

In this section we will take up the idea described in Liu (2011). This idea consists of the use of Fourier series in order to transform the approximation of the Cauchy problem into a first-kind Fredholm integral equation for the unknown function of data.

After some computation, one is brought to the search for a function f^α solution of a problem regularized by a parameter $\alpha > 0$ and whose resolution leads to an analytical expression which is a closed-form regularized solution of the initial Cauchy problem.

We begin by considering the following boundary condition instead of Eq. (8):

$$u(x,0) = f(x), \ 0 \leq x \leq L, \tag{9}$$

where $f(x)$ is an unknown function to be determined.

In other word, suppose that the datum $f(x)$ in Eq. (9) is unknown, but the Neumann datum $h(x)$ in Eq. (8) is overspecified. Determine the unknown function $f(x)$.

In this case, by the separation of variables method, solution of the problems (5)-(7) and (9) in Ω is given by

$$u(x,y) = \sum_{k=1}^{\infty} \frac{a_k \sinh[(b-y)k\pi/L]}{\sinh(bk\pi/L)} \sin(\frac{k\pi x}{L}), \tag{10}$$

where

$$a_k = \int_0^L f(t) \sin(\frac{k\pi t}{L}) dt. \tag{11}$$

The partial derivative of (10) with respect to y combined with condition (8) and Eq. (11) leads to

$$\int_0^L K(x,t) f(t) dt = h(x), \tag{12}$$

with

$$K(x,t) = \frac{2\pi}{L^2} \sum_{k=1}^{\infty} \frac{k}{\sinh(bk\pi/L)} \sin(\frac{k\pi x}{L}) \sin(\frac{k\pi t}{L}), \tag{13}$$

is a kernel function.

Thus, the solution $f(x)$ of the Cauchy problem (5)-(8), is given by solving the first-kind Fredholm integral (12). This is however a quite difficult task, since this integral equation is highly ill-posed.

Instead of (12), Liu (2011) propose to find $f(x)$ as a solution of the second-kind Fredholm integral equation

$$\alpha f(x) + \int_0^L K(x,t) f(t) dt = h(x), \tag{14}$$

where α is a Lavrentiev regularization parameter.

Its proved in that one can recover the analytically regularization solution

$$u^\alpha(x,y) = \sum_{k=1}^{\infty} a_k^\alpha \frac{\sinh[k(b-y)\pi/L]}{\sinh(kb\pi/L)} \sin(\frac{k\pi x}{L}), \tag{15}$$

with

$$a_k^\alpha = \frac{2\sinh(\frac{kb\pi}{L})}{\alpha L\sinh(\frac{kb\pi}{L}) + k\pi}\int_0^L \sin(\frac{k\pi t}{L})h(t)dt. \tag{16}$$

Replacing in (15) a_k^α by its formula given in (16) allows to write f^α and u^α using the same formula which shows an integration on $]0, L[$ and we obtain $\forall x \in [0, L]$

$$f^\alpha(x) = \frac{1}{\alpha}h(x) - \frac{2\pi}{\alpha L}\sum_{k=1}^{\infty}\frac{k\sin\left(\frac{k\pi x}{L}\right)}{\alpha L\sinh\left(\frac{bk\pi}{L}\right) + k\pi}\int_0^L \sin\left(\frac{k\pi\xi}{L}\right)h(\xi)\mathrm{d}\xi \tag{17}$$

and $\forall (x, y) \in [0, L] \times [0, b]$

$$u^\alpha(x, y) = 2\sum_{k=1}^{\infty}\frac{\sinh\left(\frac{(b-y)k\pi}{L}\right)\sin\left(\frac{k\pi x}{L}\right)}{\alpha L\sinh\left(\frac{bk\pi}{L}\right) + k\pi}\int_0^L \sin\left(\frac{k\pi\xi}{L}\right)h(\xi)\mathrm{d}\xi. \tag{18}$$

Then, equations (17) and (18) reveal one-dimensional integrals which should be approached by quadrature formulas.

3.2 Solution of the nonlinear Cauchy problem

Now an analytic regularization solution of the Cauchy problem (1) can be obtained by

$$F\left(T_\alpha\left((x, y)\right)\right) = u^\alpha\left((x, y)\right) \;\; \forall\, (x, y) \in \Omega. \tag{19}$$

Replacing u^α in (19) by its formula given in equation (18) and F by its formula given in (4), we get that $T_\alpha\left((x, y)\right)$, the analytic regularization solution of the nonlinear Cauchy problem (1) is obtained as the solution of the nonlinear equation

$$\int_0^{T_\alpha} q\left(t\right)dt - 2\sum_{k=1}^{\infty}\frac{\sinh\left(\frac{(b-y)k\pi}{L}\right)\sin\left(\frac{k\pi x}{L}\right)}{\alpha L\sinh\left(\frac{bk\pi}{L}\right) + k\pi}\int_0^L \sin\left(\frac{k\pi\xi}{L}\right)h(\xi)\mathrm{d}\xi = 0 \;\; \forall\, (x, y) \in \Omega. \tag{20}$$

We have thus reduced the search for the solution of the nonlinear Cauchy problem to the search for a solution of a nonlinear scalar equation for all $(x, y) \in \Omega$. We can prove the following results.

Theorem 1. *If there exist $\lambda > 0$ such that $\lambda \le q(t)\forall t$ and the function h is bounded on the interval $[0, L]$ then for all $\alpha > 0$ and for all $y_0 > 0$, the solution T^α converges uniformly to T for all (x, y) of $[0, L] \times [y_0, b]$ where T is the solution of nonlinear Cauchy problem (1) .*

Proof. We have, there exists ξ such that

$$F\left(T_\alpha\left((x, y)\right)\right) - F\left(T\left((x, y)\right)\right) = q(\xi)\left(T_\alpha\left((x, y)\right) - T\left((x, y)\right)\right)$$

wich imply

$$\left|F\left(T_\alpha\left((x, y)\right)\right) - F\left(T(x, y)\right)\right| \ge \lambda\left|T_\alpha\left((x, y)\right) - T\left((x, y)\right)\right|.$$

$$\begin{aligned} \left|T_\alpha\left((x, y)\right) - T\left((x, y)\right)\right| &\le \lambda^{-1}\left|F\left(T_\alpha\left((x, y)\right)\right) - F\left(T\left((x, y)\right)\right)\right| && (21)\\ &\le \lambda^{-1}\left|u_\alpha\left((x, y)\right) - u\left((x, y)\right)\right| && (22)\end{aligned}$$

Thus the uniform convergence of T_α towards T is a consequence of that of the uniform convergence of u_α towards u which is assured since h is bounded on the interval $[0, L]$, see Liu (2011). □

4 Numerical approximation

4.1 Approximation of the integral

In this context a quadrature using N_q points ζ, weighted by weights w is written:

$$\int_0^L \sin\left(\frac{k\pi\xi}{L}\right) h(\xi)\mathrm{d}\xi \approx \sum_{l=1}^{N_q} w_l \sin\left(\frac{k\pi\zeta_l}{L}\right) h(\zeta_l).$$

The key point in this quadrature approach is the trade-off between the precision and the number of points used. We opt for a Gauss-Legendre quadrature which presents, subject to a sufficient regularity of the integrand, a factorial convergence.

Note that to approach the solution in the domain Ω we have no constraint on the discretization. We can therefore use a uniform structured mesh composed of N_x+1 points in the direction x and of N_y+1 points in the direction y. The functions considered will be evaluated at the nodes (x_i, y_j) of this mesh defined as follows:

$$\forall (i,j) \in \{0,...,N_x\} \times \{0,...,N_y\}$$

$$x_i = \frac{iL}{N_x} \quad \text{and} \quad y_j = \frac{jb}{N_y}.$$

Finally, the expressions of u^α use series that have to be truncated. In practice, we will only keep m terms.

Thus, the quantity of interest u^α will be approached at the nodes (x_i, y_j) by the following discrete quantities:

$$u^\alpha(x_i, y_j) \approx 2\sum_{k=1}^{m}\sum_{l=1}^{N_q} \frac{\sinh\left(\frac{(b-y_j)k\pi}{L}\right)\sin\left(\frac{k\pi x_i}{L}\right)}{\alpha L \sinh\left(\frac{bk\pi}{L}\right) + k\pi} w_l \sin\left(\frac{k\pi\zeta_l}{L}\right) h(\zeta_l). \tag{23}$$

5 Numerical results

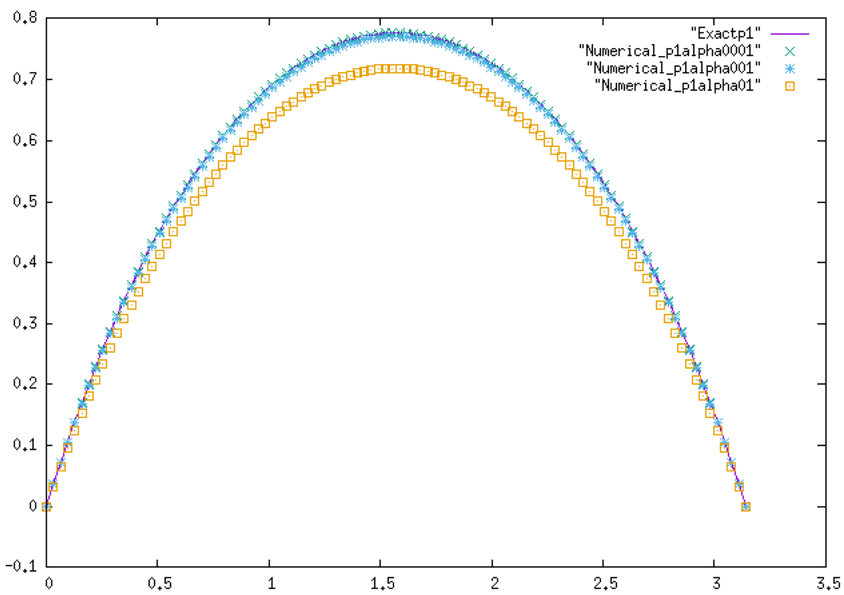

Figure 1: Comparison of the regularized and exact solution for Nonlinear Cauchy problem $p = 1$

As a typical example for testing the algorithm we take

$$\begin{cases} q(t) &= \exp(t) \\ F(t) &= \displaystyle\int_0^t \exp(u)du = \exp(t) - 1. \end{cases} \tag{24}$$

We can easily see that the exact solution of the nonlinear problem (1) is given by $T_e = ln(1+u_e)$ where u_e is the solution of the linear Cauchy problem (2). We therefore consider several examples to test our analytical solution

5.1 Example 1

We first consider a problem with $L = \pi, b = 1$ and $h(x) = \frac{\sin(px)}{p}$, where p is a positive integer. It is easily verified that the exact solution of this problem is

$$T_e(x,y) = ln(1 + \frac{\sinh(p(b-y))}{p^2}\sin(px)) \tag{25}$$

and

$$u_e(x,y) = \frac{\sinh(p(b-y))}{p^2}\sin(px) \tag{26}$$

Due to the L^2 orthogonality of $\sin(px)$ on $(0,\pi)$ the numerical solution u_α is reduced to

$$u^\alpha(x,y) = \frac{sin(px)\sinh\left[p(b-y)\right]}{\alpha p\sinh\left(bp\right) + p^2}$$

From (20), we get that T_α, the solution of the nonlinear Cauchy problem is given by

$$T_\alpha(x,y) = ln\left(1 + \frac{sin(px)\sinh\left[p(b-y)\right]}{\alpha p\sinh\left(bp\right) + p^2}\right) \tag{27}$$

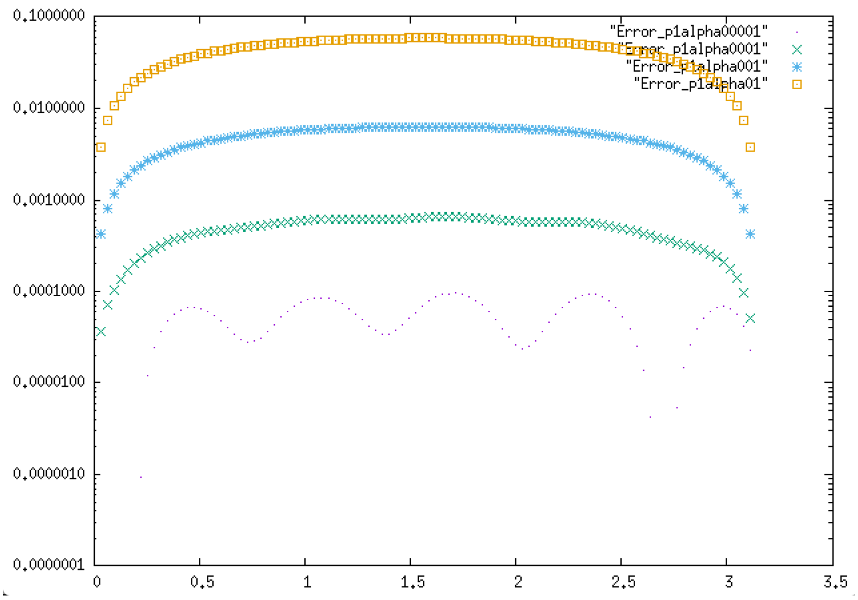

Figure 2: Comparison of the numerical errors, $p = 1$

For fixed $p = 1$, We observe from figures (1) and (2) the convergence of T_α, to T. Correspondingly, the exact solution and the numerical solution compared in Fig. (1) for a fixed $y = 0$

are, starting from $\alpha = 10^{-2}$, almost coincident. We cannot see the difference between these two solutions. The errors are plotted in Fig. (2)

It can be seen that very accurate results are obtained. The errors as can be seen are in the order of α.

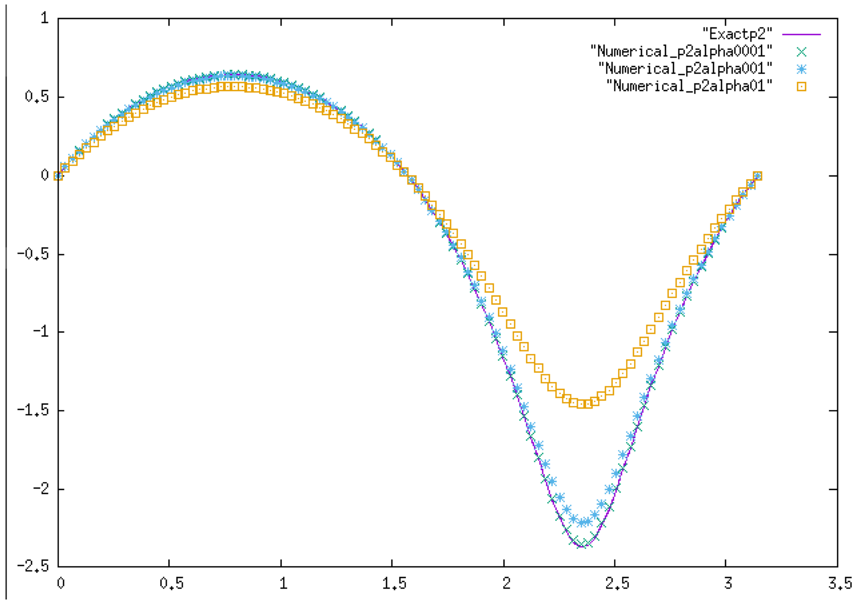

Figure 3: Comparing regularized and exact solutions for Nonlinear Cauchy problem $p = 2$

5.2 Example 2

Considering the case where the solution is more oscillating ($p = 2$), the same conclusions concerning the convergence are obtained by observing the figures

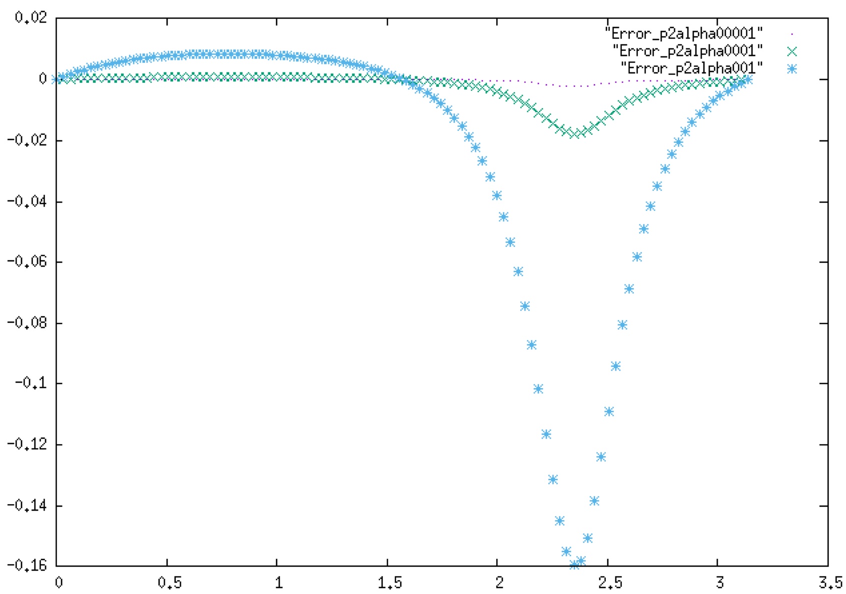

Figure 4: Comparing the numerical errors for Nonlinear Cauchy problem $p = 2$

Now, consider another example where the exact solution, is know only on Γ_1 with $L = 1$ et $b = 1$:

$$f(x) := u(x,0) = 2x\chi_{\{0\le x\le 0.5\}} + 2(1-x)\chi_{\{0.5<x\le 1\}}$$

where χ is the characteristic function. Thus , the exact solution to be recovred is $T_e(x,0) = ln(1+f(x))$.

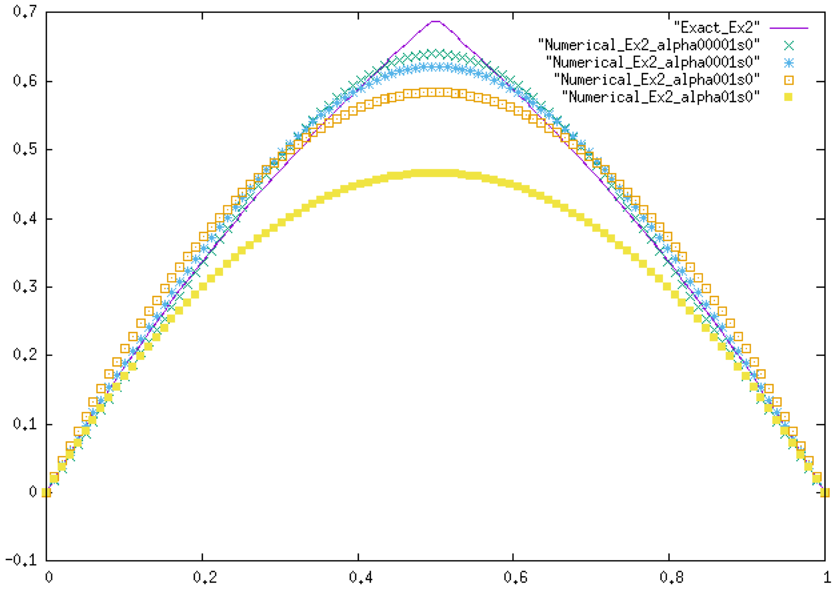

Figure 5: Comparing regularized and exact solutions without considering noise

Using this expression of f we get that $a_k = \frac{8\sin(\frac{k\pi}{2})}{k^2\pi^2}$, which itself introduced into the expression of $h(x)$ implies

$$h(x) = \pi\sum_{k=1}^{\infty}\frac{ka_k}{\sinh(bk\pi)}\sin(k\pi x). \tag{28}$$

This last expression Inserted in the one giving the form of a_k^α implies that $a_k^\alpha = \frac{k\pi a_k}{\alpha\sin(bk\pi)+k\pi}$.

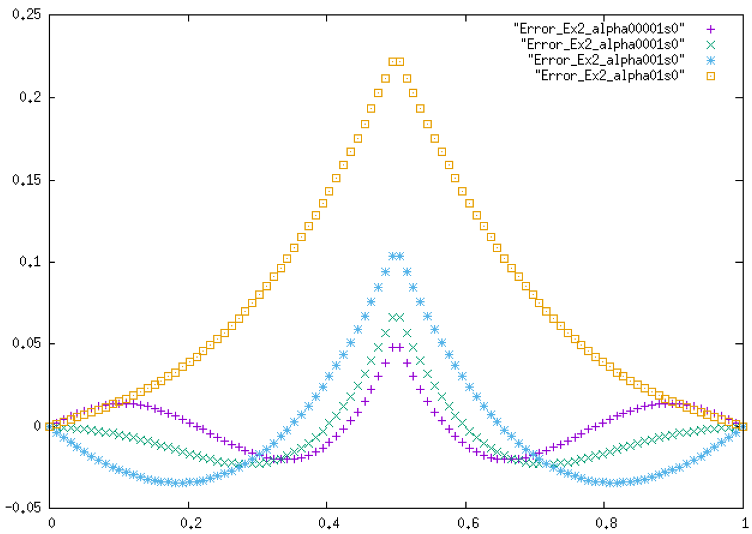

Figure 6: Comparing the numerical errors without considering noise

The numerical solution of $T_e((x,0))$ can be recovered from

$$T_e^\alpha((x,0)) = ln(1 + \sum_{k=1}^{\infty} a_k^\alpha sin(k\pi x)).$$

In Fig. (5), we plot the function $T_e^\alpha((x,0))$ for some $\alpha((x,0))$, which are compared with the exact T_e. We have calculated the above series up to 10 terms. Again, it can be seen that when α is small the numerical solutions are very close to the exact one, except that there is a little difference happened at the non-smooth point $x = 0.5$. The convergence is also confirmed in Fig (6) where the error for differente alpha is presented.

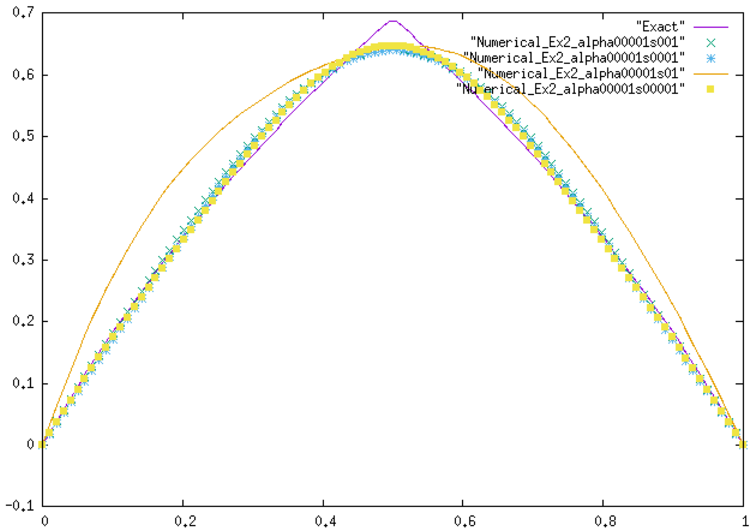

Figure 7: Comparing regularized and exact solutions with noises

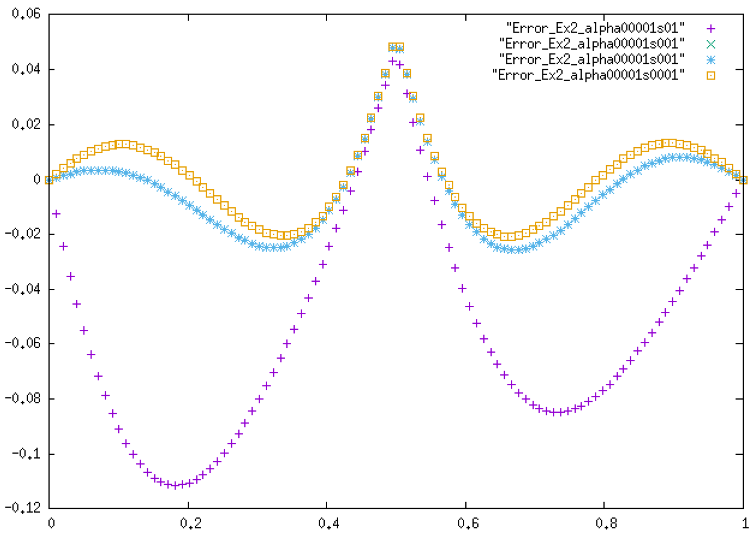

Figure 8: Comparing the numerical errors noises

To simulate the measurement errors of real boundary data we add the random noises with a zero mean and different levels of amplitude in the exact data ak defined by Eq. (64), such that the function $h(x)$ in Eq. (28) becomes

$$h(x) = \pi \sum_{k=1}^{\infty} \frac{k(a_k + sR_k)}{\sinh(bk\pi)} \sin(k\pi x),$$

where R_k are random numbers in $[-1, 1]$. The noise is obtained by multiplying R_k by a factor s.

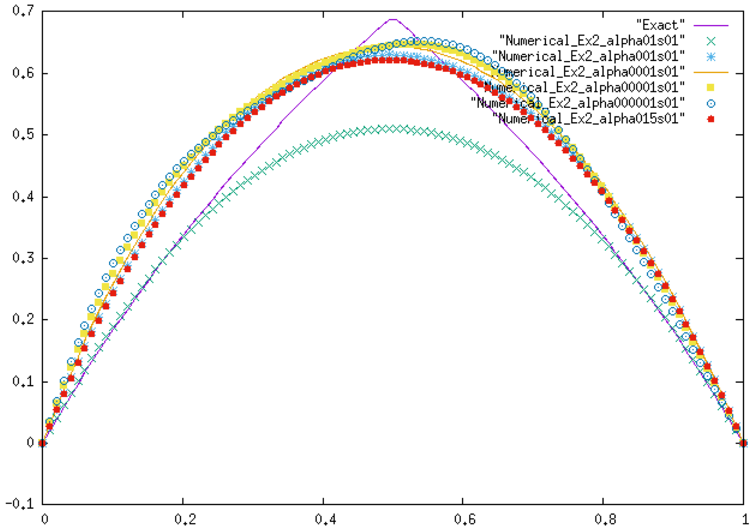

Figure 9: Comparing regularized and exact solutions with noises $s = 0.1$

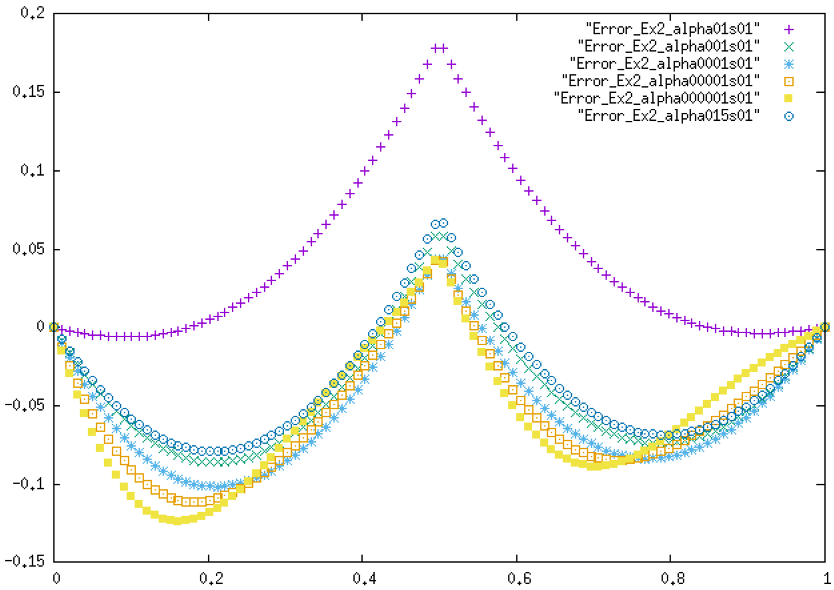

Figure 10: Comparing the numerical errors with noises $s = 0.1$

It can be seen, from figures (7) and (8) that the noises disturb the numerical solutions deviating from the exact solution quite small. Also it can be seen that the present approach is robust against the noise, even the noise is large up to 0.1.

As previously, we have represented in the figures the results where a noise $s = 0.1$ has been fixed and where we have varied α, we note that despite the high values of the noise, some values of α allow to get solutions close enough to the exact solution with fairly small errors ($\leq 5 * 10^{-2}$.

6 Conclusion

In this paper, we have presented a new analytical method to solve a nonlinear inverse Cauchy problem. For the sake of analyticity, we have limited ourselves to a simple rectangular domain for the frequent use of rectangular domain in engineering structure. The considered nonlinear problem has been transformed into a linear problem followed by a sequence of nonlinear independent scalars equations. That That allowed us to derive exactly an analytical regularization solution to approach the real solution. The solution is simply calculated and does not require any domain discretization nor linear system resolution and is therefore very efficient. The convergence of the regularization solution has been shown. Numerical examples have shown that the new method could very well recover the missing boundary data, and numerical results against noise disturbance are rather better.

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